FLOW IN A CONDUCTING FLUID OVER A POROUS PLATE IN THE PRESENCE OF A UNIFORM MAGNETIC FIELD

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The general problem of stationary, longitudinal flow of a conducting fluid over a cylindrical surface under the influence of a magnetic field and suction or blowing was investigated in [1]. As a result of an investigation of the basic equations, a class of exact solutions was found for the case where the longitudinal components of the velocity and field depend only on the coordinate normal to the surface. Several possibilities for obtaining exact solutions with a more complicated structure were also indicated. Below, an example of such a solution is constructed for the problem of flow over a porous plate.

Let us consider a porous cylindrical surface in a flow of viscous, incompressible fluid directed along the generatrix, oriented along the z-axis. Let us assume, following [1], that the velocity and magnetic field vectors have the form

$$\mathbf{V} = \mathbf{V}_\perp + v\mathbf{e}_z, \qquad \mathbf{V}_\perp = v_x(x, y)\,\mathbf{e}_x + v_y(x, y)\,\mathbf{e}_y, \qquad v = v_z(x, y)$$

$$\mathbf{H} = \mathbf{H}_\perp + h\mathbf{e}_z. \qquad \mathbf{H}_\perp = H_x(x, y)\,\mathbf{e}_x + H_y(x, y)\,\mathbf{e}_y, \qquad h = H_z(x, y, z)$$

Putting these expressions in the general equations of magnetohydrodynamics, we find, first of all, that $h = z\theta(x, y) + h_0(x, y)$ and

$$\rho(\mathbf{V}_\perp \nabla)\mathbf{V}_\perp = -\nabla' p^* + \frac{1}{4\pi}(\mathbf{H}_\perp \nabla)\mathbf{H}_\perp + \eta\triangle\mathbf{V}_\perp \qquad \left(\nabla' = \mathbf{e}_x\frac{\partial}{\partial x} + \mathbf{e}_y\frac{\partial}{\partial y}\right) \tag{1}$$

$$(\mathbf{V}_\perp \nabla)\mathbf{H}_\perp = (\mathbf{H}_\perp \nabla)\mathbf{V}_\perp + \nu_m \triangle\mathbf{H}_\perp, \qquad \operatorname{div}\mathbf{V}_\perp = 0 \tag{2}$$

$$\operatorname{div}\mathbf{H}_\perp = -\theta, \qquad \theta^2 + \mathbf{V}_\perp \triangle\theta = C \tag{3}$$

$$\mathbf{V}_\perp \nabla \theta = \nu_m \triangle \theta \tag{4}$$

$$\rho \mathbf{V}_\perp \nabla v = -P_z + \frac{1}{4\pi} \mathbf{H}_\perp \nabla h_0 + \frac{1}{4\pi} \theta h_0 + \eta \triangle v$$

$$\mathbf{V}_\perp \nabla h_0 = \mathbf{H}_\perp \nabla v - \theta v + \nu_m \triangle h_0 \tag{5}$$

The constants C and P_z are related to the gradient of the full pressure by the equation

$$\frac{\partial p^*}{\partial z} = Cz + P_z \qquad \left(p^* = p + \frac{H^2}{8\pi} \right)$$

With the first group of equations of this system, (1) to (4), together with the proper boundary conditions and the equations for the medium interior to the flow, it is possible to find the transverse velocity $\mathbf{V}_\perp$, the transverse field $\mathbf{H}_\perp$, and the function $\theta(x, y)$. Equation (4) is not independent. It may be found from the first of equations (2) by taking the operation div.

Once $\mathbf{V}_\perp$, $\mathbf{H}_\perp$ and θ are found, v and h_0 are determined from system (5). Thus the general problem separates into two, a linear and a nonlinear one, which can be solved successively. System (1) to (5) is applicable not only to external flow problems but also to problems of internal flow in tubes, under the condition that the fluid which is blown in over one part of the wall is all sucked away over another part, and the mass flow in the basic direction remains constant.

Equations (1) to (2) coincide exactly with the corresponding equations for the plane problem of magnetohydrodynamics; instead of the usual equation div $\mathbf{H}_\perp = 0$, here there are two equations (3), containing an additional unknown, scalar function θ. If $\theta \equiv 0$, then $C = 0$, and system (1) to (3) coincides entirely with the equations of the plane problem. Therefore, each solution of the latter also satisfies system (1) to (3) and generates a corresponding exact solution of system (5). This case, with $\mathbf{H}_\perp = \alpha \mathbf{V}_\perp$, α = const and rot $\mathbf{V}_\perp = 0$, is considered in [1].

The second equation of (3) is also satisfied indentically for arbitrary θ = const, different from zero. However, this does not exhaust the possible solutions of system (1) to (3). A simple exact solution with θ = const, in terms of tabulated functions, can be obtained by assuming that, over the whole flow field, the vectors $\mathbf{V}_\perp$ and $\mathbf{H}_\perp$ preserve a constant direction, for example, along the y-axis. In this case, after elimination of θ equations (1) to (3) take the form

$$\frac{\partial p^*}{\partial x} = 0, \qquad \frac{\partial p^*}{\partial y} = \frac{1}{4\pi} H_y \frac{\partial H_y}{\partial y} + \eta \frac{\partial^2 v_y}{\partial x^2}, \qquad v_y = v_y(x) \tag{6}$$

$$v_y \frac{\partial H_y}{\partial y} = \nu_m \left(\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2} \right), \qquad \left(\frac{\partial H_y}{\partial y} \right)^2 - H_y \frac{\partial^2 H_y}{\partial y^2} = C \tag{7}$$

The second equation of (7) can be integrated by quadratures, the result containing two arbitrary functions of x. Putting this result into the remaining equation, it can be shown that $v_y \equiv v_0 = \text{const}$, while H_y has one of two forms

$$H_y = H_* = \text{const}, \qquad H_y = H_0 e^{\omega y} \qquad (H_* = \text{const},\ \omega = v_0 / \nu_m) \tag{8}$$

In the first case, $\theta = C = 0$; this solution belongs to the class which was investigated in [1]. It was used in [2-3] for flows in a half-space, and in [4-6] it was applied to flow between parallel walls.* The second case corresponds to $\theta = -H_0\omega \exp \omega y$ and $C = 0$. We may note that equations (8) may also be found by another approach, by looking, as in [7], for plane magnetic fields which give rise to flows with a given structure for the velocity vector.

Let us investigate equations (5) further, for an exponentially varying magnetic field under the assumption that v and h_0 depend only on y

$$v_0 \left(\rho \frac{dv}{dy} - \frac{\alpha}{4\pi} e^{\omega y} \frac{dh_0}{dy} \right) = -P_z - \frac{v_0^2 \alpha}{4\pi \nu_m} e^{\omega y} h_0 + \eta \frac{d^2 v}{dy^2} \tag{9}$$

$$v_* \left(\frac{dh_0}{dy} - \alpha e^{\omega y} \frac{dv}{dy} \right) = \frac{v_0^2 \alpha}{\nu_m} e^{\omega y} v + \nu_m \frac{d^2 h_*}{dy^2} \qquad (\alpha = H_* / v_*) \tag{10}$$

For finding the pressure, we shall have, from (1)

$$\frac{\partial p^*}{\partial y} = \frac{v_0^3 \alpha^2}{4\pi \nu_m} e^{2\omega y}$$

Integrating (10) with respect to y, we obtain

$$\nu_m \frac{\partial h_0}{\partial y} - v_0 h_0 = cE - \alpha v_0 v e^{\omega y} \tag{11}$$

where $E = E_x = \text{const}$ is the electric field component. After elimination of $h_0' = \omega h_0$ from (9) and (11), we have

$$\frac{d^2 v}{dy^2} - \frac{v_0}{\nu} \frac{dv}{dy} - \frac{\alpha^2 v_0^2}{4\pi \nu_m \eta} v e^{2\omega y} = \frac{P_z}{\eta} - \frac{cE_1}{4\pi \nu_m} e^{\omega y} \qquad \left(E_1 = \frac{v_0 \alpha E}{\eta} \right)$$

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* The literature on this question is by no means exhausted with the cited references. Only the most recent papers are mentioned here; the necessary bibliographical information may be found in them.

or, putting $\xi = e^{\omega y}$

$$\frac{d^2v}{d\xi^2} + \frac{1-2s}{\xi}\frac{dv}{d\xi} - m^2 v = \left(\frac{1}{\omega\xi}\right)^2 \left(\frac{P_z}{\eta} - \frac{cE_1}{4\pi\nu_m}\xi\right)$$

$$\left(s = \frac{\nu_m}{2\nu}, \quad m = |\alpha| \left(\frac{s}{2\pi\rho}\right)^{1/2}\right) \tag{12}$$

The general solution of this equation has the form

$$v = \xi^s [C_1 I_s(m\xi) + C_2 K_s(m\xi)] - \frac{\xi^s}{\omega^2}\left[K_s(m\xi)\int\left(\frac{P_z}{\eta} - \frac{cE_1}{4\pi\nu_m}\xi\right) I_s(m\xi)\,\xi^{-1-s} d\xi - \right.$$
$$\left. - I_s(m\xi)\int\left(\frac{P_z}{\eta} - \frac{cE_1}{4\pi\nu_m}\xi\right) K_s(m\xi)\xi^{-1-s} d\xi\right] \tag{13}$$

Here, I_s and K_s are Bessel functions of imaginary argument.

The quantity h_0 is found from (11) by quadrature

$$h_0 = \xi\left[C_3 - \frac{cE}{v_0\xi} - \alpha\int v\,\frac{d\xi}{\xi}\right] \tag{14}$$

With formulas (13) and (14), one can investigate the boundary value problems of flow in a half-space and between parallel walls. For example, for flow in a half-space $y > 0$ with suction at the boundary ($v_0 < 0$), the constants C_1, C_2, C_3 and E can be easily related to the limiting values of v and h_0, by putting

$$\begin{aligned} &v \to v_\infty, \quad h_0 \to h_\infty \quad \text{for } y \to \infty \quad (\xi \to 0) \\ &v \to v_w, \quad h_0 \to h_w \quad \text{for } y \to 0 \quad (\xi \to 1) \end{aligned} \tag{15}$$

These conditions are satisfied by the solution

$$v = \xi^s\left\{C_1 I_s(m\xi) + C_2 K_s(m\xi) - \frac{\alpha E c}{4\pi\eta\omega}\int_\xi^1 [K_s(m\xi)\, I_s(mt) - K_s(mt)\, I_s(m\xi)]\, t^{-s}\, dt\right\} \tag{16}$$

$$h_0 = \xi\left(C_3 - \frac{cE}{v_0\xi} + \alpha\int_\xi^1 v(t)\,\frac{dt}{t}\right) \qquad \left(E = -\frac{v_0 h_\infty}{c}, \; P_z = 0\right) \tag{17}$$

$$C_1 = \frac{1}{I_s(m)}\left\{v_w - K_s(m)\left[\frac{v_\infty m^s}{2^{s-1}\Gamma(s)} - \frac{\alpha v_0 h_\infty}{4\pi\eta\omega}\int_0^1 I_s(mt)\, t^{-s}\, dt\right]\right\}$$

$$C_2 = \frac{v_\infty m^s}{2^{s-1}\Gamma(s)} - \frac{\alpha v_0 h_\infty}{4\pi\eta\omega}\int_0^1 I_s(mt)\, t^{-s}\, dt, \qquad C_3 = h_w - h_\infty$$

The shearing stress on the boundary, $\tau_w = \eta(\partial v/\partial y)_{y=0}$, corresponding to the velocity distribution (16), is expressed in the form

$$\tau_w = \frac{\rho v_0 m}{2s\, I_s(m)}\left\{v_w I_{s-1}(m) - \frac{1}{m}\left[\frac{v_\infty m^s}{2^{s-1}\Gamma(s)} - \frac{\alpha v_0 h_\infty}{4\pi\eta\omega}\int_0^1 I_s(mt)\, t^{-s}\, dt\right]\right\} \tag{18}$$

From this, we obtain for $m \ll 1$

$$\tau_w \approx \rho v_0 \left(v_w - v_\infty + \frac{\alpha v_0 h_\infty m}{8\pi\eta\omega s} \right)$$

For $m = 0$, this formula becomes the usual hydrodynamic relation, $\tau_w = \rho v_0 (v_u - v_\infty)$. If the wall is at rest and the magnetic field vanishes at infinity ($v_w = h_\infty = 0$), then, as may be seen from (18), the quantity $|\tau_w|$ decreases monotonically with increasing m.

After finding the distributions of velocity and magnetic field, the temperature field can be calculated. We start with the energy equation in the form

$$\rho c_v \left(v_0 \frac{\partial T}{\partial y} + v \frac{\partial T}{\partial z} \right) = k \triangle T + \frac{\nu_m}{4\pi} \left(\frac{\partial h}{\partial y} \right)^2 + \eta \left(\frac{\partial v}{\partial y} \right)^2 \tag{19}$$

Going over to the variables ξ and $\zeta = \omega z$, we obtain

$$\left(1 - \frac{1}{P_m}\right) \xi \frac{\partial T}{\partial \xi} + \frac{v}{v_0} \frac{\partial T}{\partial \zeta} = \frac{1}{P_m} \left(\frac{\partial^2 T}{\partial \zeta^2} + \xi^2 \frac{\partial^2 T}{\partial \xi^2} \right) + \frac{1}{4\pi\rho c_v} \left(\xi \frac{\partial h}{\partial \xi} \right)^2 + \frac{1}{2sc_v} \left(\xi \frac{dv}{d\xi} \right)^2$$

$$P_m = \frac{\rho c_v \nu_m}{k}, \qquad h = -H_0 \zeta \xi + h_0(\xi) \tag{20}$$

Here, the functions $h_0(\xi)$ and $v(\xi)$ are known from equations (16) and (17). We further denote

$$Q_0 = \frac{\xi^2}{c_v} \left(\frac{h_0'^2}{4\pi\rho} + \frac{v'^2}{2s} \right), \qquad Q_1 = -\frac{H_0}{2\pi\rho c_v} \xi h_0', \qquad Q_2 = \frac{\xi^2 H_0^2}{4\pi\rho c_v}$$

and look for a solution of (20) in the form

$$T = T_0(\xi) + \zeta T_1(\xi) + \zeta^2 T_2(\xi)$$

We obtain for the functions $T_i(\xi)$ the system

$$\left(1 - \frac{1}{P_m}\right) \xi T_2' = \frac{1}{P_m} \xi^2 T_2'' + Q_2, \qquad \left(1 - \frac{1}{P_m}\right) \xi T_1' = \frac{1}{P_m} \xi^2 T_1'' + Q_1 - \frac{2v}{v_0} T_2$$

$$\left(1 - \frac{1}{P_m}\right) \xi T_0' = \frac{1}{P_m} (\xi^2 T_0'' + 2T_2) - \frac{v}{v_0} T_1 + Q_0 \tag{21}$$

with the conditions

$$T = T_\infty \ \text{when}\ y \to \infty \ (\xi \to 0), \qquad T = T_w \ \text{when}\ y = 0 \ (\xi \to 1) \tag{22}$$

From (21) we find by quadrature

$$T_0 = T_\infty - \int_0^1 \left(\frac{vT_1}{v_0} - \frac{2T_2}{P_m} - Q_0 \right) \frac{d\xi}{\xi} + \tag{23}$$

$$+\left[T_w - T_\infty + \int_0^1 \left(\frac{vT_1}{v_0} - \frac{2T_2}{P_m} - Q_0\right)\frac{d\xi}{\xi}\right]\xi^{P_m} +$$

$$+\int_\xi^1 \left(\frac{vT_1}{v_0} - \frac{2T_2}{P_m} - Q_0\right)\frac{d\xi}{\xi} - \xi^{P_m}\int_\xi^1 \left(\frac{vT_1}{v_0} - \frac{2T_2}{P_m} - Q_0\right)\xi^{-1-P_m}\,d\xi$$

$$T_1 = (\xi^{P_m} - 1)\int_0^1 \left(\frac{2vT_2}{v_0} - Q_1\right)\frac{d\xi}{\xi} + \int_\xi^1 \left(\frac{2v\,T_2}{v_0} - Q_1\right)\frac{d\xi}{\xi} -$$

$$-\xi^{P_m}\int_\xi^1 \left(\frac{2vT_2}{v} - Q_1\right)\xi^{-1-P_m}\,d\xi\,, \qquad T_2 = \frac{H_0^2\,\nu_m}{8\pi k\,(2-P_m)}\,\xi^2\,(\xi^{P_m-2} - 1)$$

From here, the formula for heat flux at the wall is

$$q_w = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} = k\omega\,\left\{\zeta^2\,\frac{H_0^2\nu_m}{8\pi k} - \zeta P_m\left[\int_0^1\left(\frac{2vT_2}{v_0} - Q_1\right)\frac{d\xi}{\xi}\right] -\right.$$

$$\left. - P_m\left[T_w - T_\infty + \int_0^1 \left(\frac{vT_1}{v_0} - \frac{2T_2}{P_m} - Q_0\right)\frac{d\xi}{\xi}\right]\right\} \qquad (24)$$

For $v_0 \to 0$, the solution (16), (17) and (23) obtained above goes over into the solution of the problem with uniform transverse field [8], since, in that case $H_y \to$ const and $\theta \to 0$. Therefore, as may be easily shown, it is not a generalization of the results of [7], where a plane problem of the type

$$\mathbf{V} = \mathbf{e}_z v\,(y), \qquad \mathbf{H} = H_y\,(y)\,\mathbf{e}_y + (z\theta + h_0\,(y))\,\mathbf{e}_z, \qquad \theta = \text{const}$$

was investigated, notwithstanding their apparent similarity. We may note that the solution of [7] has an axisymmetric analog (see, for example, [9]), while the problem investigated above has no such analog: it is easy to prove that, for $v_r \neq 0$ and $\partial H_z/\partial z \neq 0$, the magnetohydrodynamic equations do not allow a solution of the form

$$\mathbf{V} = v_r\,(r)\,\mathbf{e}_r + v_z\,(r)\,\mathbf{e}_z, \qquad \mathbf{H} = H_r\,(r)\,\mathbf{e}_r + H_\theta\,(r)\,\mathbf{e}_\theta + H_z\,(r, z)\,\mathbf{e}_z$$

It should be stated, in closing, that the practical realization of the situation described by the obtained solution is very difficult.

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